

# VARIETIES FOR MODULES OF FINITE DIMENSIONAL HOPF ALGEBRAS

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*Dedicated to Professor David J. Benson on the occasion of his 60th birthday.*

**ABSTRACT.** We survey variety theory for modules of finite dimensional Hopf algebras, recalling some definitions and basic properties of support and rank varieties where they are known. We focus specifically on properties known for classes of examples such as finite group algebras and finite group schemes. We list open questions about tensor products of modules and projectivity, where varieties may play a role in finding answers.

## 1. INTRODUCTION

For a given group or ring, one wants to understand its representations in a meaningful way. It is often too much to ask for a full classification of all indecomposable modules, since one may work in a setting of wild representation type. Varieties can then be an important tool for organizing representations and extracting information. In the theory of varieties for modules, one associates to each module a geometric space—typically an affine or projective variety—in such a way that representation theoretic properties are encoded in the space. Varieties for modules originated in finite group representation theory, in work of Quillen [43] and Carlson [16]. This theory and all required background material is elegantly presented in Benson’s book [7]. The theory has been adapted to many other settings, such as finite group schemes, algebraic groups, Lie superalgebras, quantum groups, and self-injective algebras. See, e.g., [1, 2, 15, 18, 19, 24, 26, 34, 37, 38, 51, 47].

In this survey article, we focus on finite dimensional Hopf algebras, exploring the boundary between those whose variety theory behaves as one expects, arriving from finite group representation theory, and those where it does not. We give definitions of support varieties in terms of Hochschild cohomology from [19, 47], and in terms of Hopf algebra cohomology as a direct generalization of group cohomology from [23, 27, 28, 37]. We recall which Hopf algebras are known to have finitely generated cohomology, opening the door to these standard versions of support

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*Date:* 2 December 2016.

The author was partially supported by NSF grant #DMS-1401016.

varieties. We also briefly summarize the rank varieties which are defined representation theoretically in a more limited array of settings, yet are indispensable where they are defined.

We are most interested in the tensor product property, that is, the property that the variety of a tensor product of modules is equal to the intersection of their varieties. This is known to hold for modules of some Hopf algebras, known not to hold for others, and is an open question for most. We look at some related questions about tensor products of modules: (i) If the tensor product of two modules in one order is projective, what about their tensor product in the other order? (ii) If a tensor power of a module is projective, need the module itself be projective? The answers to both questions are yes for finite group algebras and finite group schemes, while the answers to both are no for some types of Hopf algebras, as we will see. In fact, any finite dimensional Hopf algebra satisfying the tensor product property is a subalgebra of one that does not and for which the above two questions have negative answers.

The open questions we discuss in this article are important for gaining a better understanding of the representation theory of finite dimensional Hopf algebras. Their module categories enjoy a rich structure due to existence of tensor products. Varieties are a great tool for understanding these tensor products when one understands the relationship between them and their varieties.

Throughout, we will work over an algebraically closed field  $k$ , although there are known results for more general fields and ground rings in some contexts. Sometimes we will assume  $k$  has positive characteristic, and sometimes that it has characteristic 0. All tensor products will be taken over  $k$  unless otherwise indicated, that is,  $\otimes = \otimes_k$ . All modules will be left modules, finite dimensional over  $k$ , unless otherwise stated.

## 2. HOPF ALGEBRAS

A *Hopf algebra* is an algebra  $A$  over the field  $k$  together with  $k$ -linear maps  $\Delta : A \rightarrow A \otimes A$  (comultiplication),  $\varepsilon : A \rightarrow k$  (counit or augmentation), and  $S : A \rightarrow A$  (antipode or coinverse) satisfying the following properties: The maps  $\Delta$  and  $\varepsilon$  are algebra homomorphisms, and  $S$  is an algebra anti-homomorphism (i.e., it reverses the order of multiplication). Symbolically writing  $\Delta(a) = \sum a_1 \otimes a_2$  (Sweedler notation), we also require

$$\begin{aligned} (1 \otimes \Delta)(\Delta(a)) &= (\Delta \otimes 1)(\Delta(a)) \quad (\text{coassociativity}), \\ \sum \varepsilon(a_1)a_2 &= a = \sum a_1\varepsilon(a_2) \quad (\text{counit property}), \\ \sum S(a_1)a_2 &= \varepsilon(a) \cdot 1 = \sum a_1S(a_2) \quad (\text{antipode property}) \end{aligned}$$

for all  $a \in A$ . We say that  $A$  is *cocommutative* if  $\tau \circ \Delta = \Delta$ , where  $\tau : A \otimes A \rightarrow A \otimes A$  is the twist map, that is,  $\tau(a \otimes b) = b \otimes a$  for all  $a, b \in A$ . For more details, see, e.g., [35].

Standard examples of Hopf algebras, some of which will reappear in later sections, are:

**Example 2.1.**  $A = kG$ , the group algebra of a finite group  $G$ , with  $\Delta(g) = g \otimes g$ ,  $\varepsilon(g) = 1$ , and  $S(g) = g^{-1}$  for all  $g \in G$ . This Hopf algebra is cocommutative.

**Example 2.2.**  $A = k[G] = \text{Hom}_k(kG, k)$ , the linear dual of the group algebra  $kG$ , in which multiplication is pointwise on group elements, that is,  $(ff')(g) = f(g)f'(g)$  for all  $g \in G$  and  $f, f' \in k[G]$ . Comultiplication is given as follows. Let  $\{p_g \mid g \in G\}$  be the basis of  $k[G]$  dual to  $G$ . Then

$$\Delta(p_g) = \sum_{\substack{a, b \in G \\ ab=g}} p_a \otimes p_b,$$

$\varepsilon(p_g) = \delta_{g,1}$ , and  $S(p_g) = p_{g^{-1}}$  for all  $g \in G$ . This Hopf algebra is noncocommutative when  $G$  is nonabelian.

**Example 2.3.**  $A = U(\mathfrak{g})$ , the universal enveloping algebra of a Lie algebra  $\mathfrak{g}$ , with  $\Delta(x) = x \otimes 1 + 1 \otimes x$ ,  $\varepsilon(x) = 0$ , and  $S(x) = -x$  for all  $x \in \mathfrak{g}$ . The maps  $\Delta$  and  $\varepsilon$  are extended to be algebra homomorphisms, and  $S$  to be an algebra anti-homomorphism. This is an infinite dimensional cocommutative Hopf algebra. In case the characteristic of  $k$  is a prime  $p$ , and  $\mathfrak{g}$  is a restricted Lie algebra, its restricted enveloping algebra  $u(\mathfrak{g})$  is a finite dimensional cocommutative Hopf algebra with analogous comultiplication, counit, and antipode.

**Example 2.4.**  $A = U_q(\mathfrak{g})$  or  $A = u_q(\mathfrak{g})$ , the infinite dimensional quantum enveloping algebras and some finite dimensional versions (the small quantum groups). See, e.g., [30] for the definition in the general case. Here we give just one small example explicitly: Let  $q$  be a primitive complex  $n$ th root of unity,  $n > 2$ . Let  $u_q(\mathfrak{sl}_2)$  be the  $\mathbb{C}$ -algebra generated by  $E, F, K$  with  $E^n = 0$ ,  $F^n = 0$ ,  $K^n = 1$ ,  $KE = q^2 EK$ ,  $KF = q^{-2} FK$ , and

$$EF = FE + \frac{K - K^{-1}}{q - q^{-1}}.$$

Let  $\Delta(E) = E \otimes 1 + K \otimes E$ ,  $\Delta(F) = F \otimes K^{-1} + 1 \otimes F$ ,  $\Delta(K) = K \otimes K$ ,  $\varepsilon(E) = 0$ ,  $\varepsilon(F) = 0$ ,  $\varepsilon(K) = 1$ ,  $S(E) = -K^{-1}E$ ,  $S(F) = -FK$ , and  $S(K) = K^{-1}$ . This is a finite dimensional noncocommutative Hopf algebra.

**Example 2.5.**  $A$  is a quantum elementary abelian group: Let  $m$  and  $n$  be positive integers,  $n \geq 2$ . Let  $q$  be a primitive complex  $n$ th root of unity, and let  $A$  be the  $\mathbb{C}$ -algebra generated by  $x_1, \dots, x_m, g_1, \dots, g_m$  with relations  $x_i^n = 0$ ,  $g_i^n = 1$ ,  $x_i x_j = x_j x_i$ ,  $g_i g_j = g_j g_i$ , and  $g_i x_j = q^{\delta_{i,j}} x_j g_i$  for all  $i, j$ . Comultiplication is given by  $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$ ,  $\Delta(g_i) = g_i \otimes g_i$ , counit  $\varepsilon(x_i) = 0$ ,  $\varepsilon(g_i) = 1$ , and antipode

$S(x_i) = -g_i^{-1}x_i$ ,  $S(g_i) = g_i^{-1}$  for all  $i$ . This Hopf algebra is finite dimensional and noncocommutative.

We return to the general setting of a Hopf algebra  $A$ , which we assume from now on is finite dimensional over  $k$ . Letting  $M$  and  $N$  be  $A$ -modules, their tensor product  $M \otimes N$  is again an  $A$ -module via the comultiplication map  $\Delta$ , that is,  $a \cdot (m \otimes n) = \sum (a_1 \cdot m) \otimes (a_2 \cdot n)$  for all  $a \in A$ ,  $m \in M$ ,  $n \in N$ . The category of (finite dimensional)  $A$ -modules is a rigid tensor category: There is a unit object given by the field  $k$  under action via the counit  $\varepsilon$ , i.e.,  $a \cdot c = \varepsilon(a)c$  for all  $a \in A$  and  $c \in k$ . There are dual objects: Let  $M$  be a finite dimensional  $A$ -module, and let  $M^* = \text{Hom}_k(M, k)$ , an  $A$ -module via  $S$ :  $(a \cdot f)(m) = f(S(a) \cdot m)$  for all  $a \in A$ ,  $m \in M$ . See, e.g., [4] for details on rigid tensor categories.

The following proposition is proven in [6, Proposition 3.1.5]. An alternative proof is to observe that  $\text{Hom}_A(P, \text{Hom}_k(M, -)) \cong \text{Hom}_A(P \otimes M, -)$  as functors, where the action of  $A$  on  $\text{Hom}_k(M, N)$ , for  $A$ -modules  $M, N$ , is given by  $(a \cdot f)(m) = \sum a_1 \cdot (f(S(a_2) \cdot m))$  for all  $a \in A$ ,  $m \in M$ , and  $f \in \text{Hom}_k(M, N)$ . (A similar argument applies to  $M \otimes P$ .)

**Proposition 2.6.** *If  $P$  is a projective  $A$ -module, and  $M$  is any  $A$ -module, then both  $P \otimes M$  and  $M \otimes P$  are projective  $A$ -modules.*

In Section 4, we will consider other connections between projectivity and tensor products.

### 3. VARIETIES FOR MODULES

There are many versions of varieties for modules, depending on which rings and modules are of interest. Here we will present the support variety theory of Erdmann, Holloway, Snashall, Solberg, and Taillefer [19, 47] for self-injective algebras (based on Hochschild cohomology), as well as the closely related generalization to Hopf algebras of support varieties for finite group representations (see, e.g., [23, 25, 27, 37]). See also Solberg's excellent survey [48] for more details.

**Hochschild cohomology.** Let  $A$  be an associative  $k$ -algebra. Let  $A^e = A \otimes A^{op}$ , with  $A^{op}$  the opposite algebra to  $A$ . Consider  $A$  to be an  $A^e$ -module via left and right multiplication, that is,  $(a \otimes b) \cdot c = acb$  for all  $a, b, c \in A$ . The *Hochschild cohomology* of  $A$  is

$$\text{HH}^*(A) = \text{Ext}_{A^e}^*(A, A).$$

The graded vector space  $\text{HH}^*(A)$  is a graded commutative ring under Yoneda composition/cup product [7, 50], and  $\text{HH}^0(A) \cong Z(A)$ , the center of  $A$ . If  $M$  is an  $A$ -module, then the Hochschild cohomology ring  $\text{HH}^*(A)$  acts on  $\text{Ext}_A^*(M, M)$  via  $- \otimes_A M$  followed by Yoneda composition.

Now suppose  $A$  is finite dimensional and self-injective. For example, a finite dimensional Hopf algebra is a Frobenius algebra, and therefore is self-injective. We will make some assumptions, as in [19]:

Assume there is a graded subalgebra  $H$  of  $\mathrm{HH}^*(A)$  such that

- (fg1)  $H$  is finitely generated, commutative, and  $H^0 = \mathrm{HH}^0(A)$ , and
- (fg2) for all finite dimensional  $A$ -modules  $M$ , the Ext space  $\mathrm{Ext}_A^*(M, M)$  is finitely generated as an  $H$ -module.

For a finite dimensional  $A$ -module  $M$ , let  $I_A(M)$  be the annihilator in  $H$  of  $\mathrm{Ext}_A^*(M, M)$ . The *support variety* of  $M$  is

$$(3.1) \quad V_A(M) = \mathrm{Max}(H/I_A(M)),$$

the maximal ideal spectrum of  $H/I_A(M)$ . This is the set of maximal ideals as a topological space under the Zariski topology. Alternatively, one considers homogeneous prime ideals as in some of the given references. Here we choose maximal ideals as in [19].

**Example 3.2.** Let  $A = kG$ , where  $G$  is a finite group. We assume the characteristic of  $k$  is a prime  $p$  dividing the order of  $G$ , since otherwise  $kG$  is semisimple by Maschke's Theorem. The *group cohomology* ring is  $H^*(G, k) = \mathrm{Ext}_{kG}^*(k, k)$ . More generally if  $M$  is a  $kG$ -module, set  $H^*(G, M) = \mathrm{Ext}_{kG}^*(k, M)$ . There is an algebra isomorphism:

$$\mathrm{HH}^*(kG) \cong H^*(G, (kG)^{ad})$$

(see, e.g., [45, Proposition 3.1]), where the latter is group cohomology with coefficients in the adjoint  $kG$ -module  $kG$  (on which  $G$  acts by conjugation). The group cohomology  $H^*(G, k)$  then embeds into Hochschild cohomology  $\mathrm{HH}^*(kG)$ , since the trivial coefficients  $k \cdot 1$  embed as a direct summand of  $(kG)^{ad}$ . Let

$$H = H^{ev}(G, k) \cdot \mathrm{HH}^0(kG),$$

where  $H^{ev}(G, k)$  is  $H^*(G, k)$  if  $\mathrm{char}(k) = 2$  and otherwise is the subalgebra of  $H^*(G, k)$  generated by its homogeneous even degree elements, considered to be a subalgebra of Hochschild cohomology  $\mathrm{HH}^*(kG)$  via the embedding discussed above. Then  $H$  satisfies (fg1) and (fg2). The traditional definition of varieties for  $kG$ -modules uses simply  $H^{ev}(G, k)$  instead of  $H$ , the difference being the inclusion of the elements of  $\mathrm{HH}^0(kG) \cong Z(kG)$ . If  $G$  is a  $p$ -group, there is no difference in the theories since  $Z(kG)$  is local. See, e.g., [7] for details, including descriptions of the original work of Golod [31], Venkov [52], and Evens [21] on finite generation. If  $G$  is not a  $p$ -group, the representation theoretic information contained in the varieties will be largely the same in the two cases (the only exception being the additional information of which block(s) a module lies in).

Returning to the general setting of a finite dimensional self-injective algebra  $A$ , the support varieties defined above enjoy many useful properties [19], some of

which we collect below. We will need to define the complexity of a module: The *complexity*  $\text{cx}_A(M)$  of a finite dimensional  $A$ -module  $M$  is the rate of growth of a minimal projective resolution. That is, if  $P_\bullet$  is a minimal projective resolution of  $M$ , then  $\text{cx}_A(M)$  is the smallest nonnegative integer  $c$  such that there is a real number  $b$  and positive integer  $m$  for which  $\dim_k(P_n) \leq bn^{c-1}$  for all  $n \geq m$ . A projective module has complexity 0. The converse is also true, as stated in the following proposition.

**Proposition 3.3.** [19, 47] *Let  $A$  be a finite dimensional self-injective algebra for which there is an algebra  $H$  satisfying (fg1) and (fg2). Let  $M$  and  $N$  be finite dimensional  $A$ -modules. Then:*

- (i)  $\dim V_A(M) = \text{cx}_A(M)$ .
- (ii)  $V_A(M \oplus N) = V_A(M) \cup V_A(N)$ .

Moreover,  $\dim V_A(M) = 0$  if and only if  $M$  is projective.

We will apply the support variety theory of [19, 47], as outlined above, to a finite dimensional Hopf algebra  $A$ , provided there exists an algebra  $H$  satisfying (fg1) and (fg2).

**Hopf algebra cohomology.** Alternatively, one may generalize support varieties for finite groups directly: The *cohomology* of the Hopf algebra  $A$  is

$$H^*(A, k) = \text{Ext}_A^*(k, k).$$

The cohomology  $H^*(A, k)$  is a graded commutative ring under Yoneda composition/cup product [50]. If  $M$  is an  $A$ -module, consider  $\text{Ext}_A^*(M, M)$  to be an  $H^*(A, k)$ -module via  $- \otimes M$  followed by Yoneda composition. We make the following assumptions, as in [23]:

Assume that

- (fg1')  $H^*(A, k)$  is a finitely generated algebra, and
- (fg2') for all finite dimensional  $A$ -modules  $M$ , the Ext space  $\text{Ext}_A^*(M, M)$  is finitely generated as a module over  $H^*(A, k)$ .

Then one defines the *support variety* of an  $A$ -module  $M$  to be the maximal ideal spectrum of the quotient of  $H^*(A, k)$  by the annihilator of  $\text{Ext}_A^*(M, M)$ . By abuse of notation, we will also write  $V_A(M)$  for this variety, and in the sequel it will be clear in each context which is meant. If one wishes to work with a commutative ring from the beginning, and not just a graded commutative ring, then in characteristic not 2, one first restricts to the subalgebra  $H^{ev}(A, k)$  of  $H^*(A, k)$  generated by all homogeneous elements of even degree. (The odd degree elements are nilpotent, and so the varieties are the same.) Proposition 3.3 holds for these varieties [23].

There is a close connection between this version of support variety and that defined earlier via Hochschild cohomology: Just as in Example 3.2, Hopf algebra cohomology  $H^*(A, k)$  embeds into Hochschild cohomology  $\text{HH}^*(A)$ . See, e.g., [30]

where this fact was first noted and the appendix of [40] for a proof outline. One may then take  $H = H^{ev}(A, k) \cdot HH^0(A)$  in order to define support varieties as in (3.1). The only difference between these two versions of support variety is the inclusion of the elements of  $HH^0(A) \cong Z(A)$ . Thus there is a finite surjective map from the variety defined via the subalgebra  $H$  of Hochschild cohomology to the variety defined via Hopf algebra cohomology.

**Rank varieties.** We now consider the rank varieties that were first introduced by Carlson [16] for studying finite group representations. We recall his definition and discuss some Hopf algebras for which there are analogs. Carlson's rank varieties are defined for elementary abelian  $p$ -groups. For a finite group  $G$ , its elementary abelian  $p$ -subgroups detect projectivity by Chouinard's Theorem [7, Theorem 5.2.4], and form the foundation for stratification of support varieties [3, 43]. Thus it is important to understand the elementary abelian  $p$ -subgroups of  $G$  and their rank varieties as defined below.

Suppose  $k$  is a field of prime characteristic  $p$ . An *elementary abelian  $p$ -group* is a group of the form  $E = (\mathbb{Z}/p\mathbb{Z})^n$  for some  $n$ . Write  $E = \langle g_1, \dots, g_n \rangle$ , where  $g_i$  generates the  $i$ th copy of  $\mathbb{Z}/p\mathbb{Z}$  in  $E$ . For each  $i$ , let  $x_i = g_i - 1$ , and note that  $x_i^p = 0$  since  $\text{char}(k) = p$  and  $g_i^p = 1$ . It also follows that any element of the group algebra  $kE$  of the form  $\lambda_1 x_1 + \dots + \lambda_n x_n$  ( $\lambda_i \in k$ ) has  $p$ th power 0. Thus for each choice of scalars  $\lambda_1, \dots, \lambda_n$ , there is an algebra homomorphism

$$\begin{aligned} k[t]/(t^p) &\rightarrow kE \\ t &\mapsto \lambda_1 x_1 + \dots + \lambda_n x_n. \end{aligned}$$

The image of this homomorphism is a subalgebra of  $kE$  that we will denote by  $k\langle \lambda_1 x_1 + \dots + \lambda_n x_n \rangle$ . Note that it is isomorphic to  $k\mathbb{Z}/p\mathbb{Z}$  where the group  $\mathbb{Z}/p\mathbb{Z}$  is generated by  $1 + \lambda_1 x_1 + \dots + \lambda_n x_n$ . The corresponding subgroup of the group algebra  $kE$  is called a *cyclic shifted subgroup* of  $E$ . The *rank variety* of a  $kE$ -module  $M$  is

$$V_E^r(M) = \{\mathbf{0}\} \cup \{(\lambda_1, \dots, \lambda_n) \in k^n - \{\mathbf{0}\} \mid M \downarrow_{k\langle \lambda_1 x_1 + \dots + \lambda_n x_n \rangle} \text{ is not free}\},$$

where the downarrow indicates restriction to the subalgebra. Avrunin and Scott [3] proved that the rank variety  $V_E^r(M)$  is homeomorphic to the support variety  $V_E(M)$  (which we have also denoted  $V_{kE}(M)$ ). Information about a more general finite group  $G$  is obtained by looking at all its elementary abelian  $p$ -subgroups. It is very useful to have on hand these rank varieties for modules, as another way to view the support varieties.

Friedlander and Pevtsova [28] generalized rank varieties to finite dimensional cocommutative Hopf algebras  $A$  (equivalently finite group schemes), building on earlier work of Friedlander and Parshall [25] and Suslin, Friedlander, and Bendel [51]. The role of cyclic shifted subgroups is played by subalgebras isomorphic to  $k[t]/(t^p)$ , or more generally by algebras  $K[t]/(t^p)$  for field extensions  $K$  of  $k$ ,

and specific types of maps to extensions  $A_K$ . A notion of rank variety for quantum elementary abelian groups is defined in [40], where the role of cyclic shifted subgroups is played by subalgebras isomorphic to  $k[t]/(t^n)$  with  $n$  the order of the root of unity  $q$ . Scherotzke and Towers [44] defined rank varieties for  $u_q(\mathfrak{sl}_2)$ , and for the related Drinfeld doubles of Taft algebras, via certain subalgebras detecting projectivity. Rank varieties have been defined as well for a number of algebras that are not Hopf algebras; see, e.g., [9, 10, 14]. In general though, it is not always clear what the right definition of rank variety should be, if any.

#### 4. OPEN QUESTIONS AND SOME POSITIVE ANSWERS

We next ask some questions about finite dimensional Hopf algebras, their representations, and varieties. We refer to the previous section for descriptions of the support and rank varieties relevant to Question 4.1(2) below. We have purposely not specified choices of varieties for the question, and answers may depend on choices. However, answers to the purely representation theoretic Questions 4.1(3) and (4) below do not.

**Questions 4.1.** *Let  $A$  be a finite dimensional Hopf algebra.*

- (1) *Does  $H^*(A, k)$  satisfy  $(fg1')$  and  $(fg2')$ , or does there exist a subalgebra  $H$  of  $HH^*(A)$  satisfying  $(fg1)$  and  $(fg2)$ ?*

If the answer to (1) is yes, or if one has at hand a version of rank varieties or other varieties for  $A$ -modules, one may further ask:

- (2) *Is  $V_A(M \otimes N) = V_A(M) \cap V_A(N)$  for all finite dimensional  $A$ -modules  $M, N$ ?*

The property in (2) above is called the *tensor product property* of varieties for modules. The following questions may be asked independently of the first two.

- (3) *For all finite dimensional  $A$ -modules  $M, N$ , is  $M \otimes N$  projective if and only if  $N \otimes M$  is projective?*
- (4) *For all finite dimensional  $A$ -modules  $M$  and positive integers  $n$ , is  $M$  projective if and only if  $M^{\otimes n}$  is projective?*

Note that for a given Hopf algebra  $A$ , if the answers to Questions 4.1(1) and (2) are yes, then the answers to (3) and (4) are yes: By the tensor product property,  $V_A(M \otimes N) = V_A(N \otimes M)$ , and by Proposition 3.3, this variety has dimension 0 if and only if  $M \otimes N$  (respectively  $N \otimes M$ ) is projective. Also by the tensor product property,  $V_A(M^{\otimes n}) = V_A(M)$ , and again  $M^{\otimes n}$  (respectively  $M$ ) is projective if and only if the dimension of its support variety is 0.

We will see in the next section that there are Hopf algebras for which the answer to Question 4.1(2) is no, and yet there is another way to express  $V_A(M \otimes N)$  in terms of  $V_A(M)$  and  $V_A(N)$ . So we may wish to consider instead one of the following questions about all finite dimensional  $A$ -modules  $M, N$ :



(2') Can  $V_A(M \otimes N)$  be expressed in terms of  $V_A(M)$  and  $V_A(N)$ ?  
 or (2'') Is  $\dim V_A(M \otimes N) = \dim(V_A(M) \cap V_A(N))$ ?

For either of these questions, if the answer is yes, one may still use support varieties to obtain valuable information about the tensor product structure of modules, for example, the property in (2'') allows us to understand the complexity of  $M \otimes N$  using knowledge of the support varieties of the tensor factors  $M, N$ .

Many mathematicians have worked on Question 4.1(1). It is closely related to a conjecture of Etingof and Ostrik [20] that the cohomology ring of a finite tensor category is finitely generated; this includes the category of finite dimensional modules of a finite dimensional Hopf algebra as a special case. This is condition (fg1'). The further condition (fg2') should follow using similar proof techniques as for the finite generation of  $H^*(A, k)$ . One can then take  $H$  to be  $H^{ev}(A, k) \cdot HH^0(A)$ , or use  $H^*(A, k)$  directly to define support varieties, as explained in Section 3. As a cautionary note however, a related conjecture about Hochschild cohomology of finite dimensional algebras was shown to be false; see, e.g., [46, 47, 53].

We next discuss some general classes of Hopf algebras for which the answers to all four Questions 4.1 are known to be yes, as well as those for which some of the four questions are known to have positive answers, while others remain open. In the next section, we discuss some classes of Hopf algebras for which the answer to at least one of the four questions is no.

**Finite group algebras.** If  $A = kG$ , where  $G$  is a finite  $p$ -group and  $\text{char}(k) = p$ , the answers to all four Questions 4.1 are yes: As explained in Example 3.2, one may take  $H = H^{ev}(G, k) \cdot HH^0(kG)$ . Since  $HH^0(kG) \cong Z(kG)$  is a local ring, for the purpose of defining varieties, this is equivalent to taking  $H$  to be simply  $H^{ev}(G, k)$ , the standard choice (see, e.g., [7, 16]). If  $G$  is not a  $p$ -group, the standard version of varieties for modules and the one coming from Hochschild cohomology differ by finite surjective maps. The answer to Question 4.1(2) is yes for the standard version and is no for the Hochschild cohomology version (tensor a module in a nonprincipal block with the trivial module  $k$ ). However the answer to the modified question (2') is still yes in this case. The answers to Questions 4.1(3) and (4) are yes.

**Finite group schemes.** If  $A$  is a finite dimensional cocommutative Hopf algebra (equivalently, finite group scheme), the answers are also known due to work of many mathematicians building on work on finite groups, on restricted Lie algebras [25], and on infinitesimal group schemes [51]. See, e.g., [25, 28, 29, 51], and the surveys [24, 39]. In this case, one works with support varieties defined via Hopf algebra cohomology  $H^*(A, k)$ , which is known to satisfy conditions (fg1') and (fg2') [25, 29], and with rank varieties, which are homeomorphic to the support varieties. The tensor product property is proven using rank varieties [28]. The answers to all four Questions 4.1 are yes, under these choices.

**Quantum elementary abelian groups.** If  $A$  is a quantum elementary abelian group as defined in Example 2.5, the answers to Questions 4.1 are known. Again we work with support varieties defined via Hopf algebra cohomology  $H^*(A, k)$  which satisfies (fg1') and (fg2') [40]. The tensor product property is proven using rank varieties [41], and the answers to all four Questions 4.1 are yes.

**Finite quantum groups and function algebras and more.** If  $A$  is a small quantum group  $u_q(\mathfrak{g})$  (see Example 2.4), by [5, 30], the cohomology  $H^*(A, k)$  is known to be finitely generated for most values of the parameters, and the answer to Question 4.1(1) is yes. However Question 4.1(2) is open; it was conjectured by Ostrik [37] who developed support variety theory for these Hopf algebras. Likewise for some more general pointed Hopf algebras with abelian groups of grouplike elements (see [36]) and some finite quotients of quantum function algebras (see [32]). If  $A$  is the 12-dimensional Fomin-Kirillov algebra—a pointed Hopf algebra with non-abelian group of grouplike elements—then  $H^*(A, k)$  is finitely generated (see [49]). For all of these important examples, there should be a good support variety theory, yet the tensor product property is unknown.

Applications of varieties for modules abound, and are well developed for some of the classes of Hopf algebras described above. For example, one can construct modules with prescribed support (see [2, 7, 23, 28]). Representation type can be seen in the varieties (see [22, 23, 33]). Some of the structure of the (stable) module category can be understood from knowledge of particular subcategories analogous to ideals in a ring, and these are typically parametrized by support varieties (see [8, 11, 12, 13, 17, 41]).

Most of the foregoing discussion focuses on Questions 4.1(1) and (2). We now give a general context in which the answer to Question 4.1(4) is known to be yes, independently of any variety theory: Let  $A$  be a finite dimensional Hopf algebra, let  $M$  be an  $A$ -module for which  $M \otimes M^* \cong M^* \otimes M$ , and let  $n$  be a positive integer. Then  $M$  is projective if and only if  $M^{\otimes n}$  is projective. This statement is a consequence of rigidity, since rigidity implies that  $M$  is a direct summand of  $M \otimes M^* \otimes M \cong M^* \otimes M \otimes M$ . See [42] for details. Hopf algebras for which the tensor product of modules is commutative up to isomorphism (such as almost cocommutative or quasitriangular Hopf algebras) always satisfy this condition, and so the answer to Question 4.1(4) is yes for these Hopf algebras.

## 5. SOME NEGATIVE ANSWERS

In this section we give examples of Hopf algebras for which the answer to Question 4.1(1) is yes, and there is a reasonable support variety theory for which the answer to Question 4.1(2') is yes, however the answers to Questions 4.1(2), (2''), (3), and (4) are no.

Our first class of examples is from [13] with Benson. Let  $k$  be a field of positive characteristic  $p$  and let  $L$  be a finite  $p$ -group. Let  $G$  be a finite group acting on  $L$  by automorphisms. Let

$$A = kL \otimes k[G]$$

as an algebra, where  $k[G]$  is the linear dual of the group algebra  $kG$  as in Example 2.2. The comultiplication is not the tensor product comultiplication, but rather is modified by the group action:

$$\Delta(x \otimes p_g) = \sum_{\substack{a, b \in G \\ ab=g}} (x \otimes p_a) \otimes ((a^{-1} \cdot x) \otimes p_b)$$

for all  $x \in L$ ,  $g \in G$ . The counit and antipode are given by  $\varepsilon(x \otimes p_g) = \delta_{g,1}$  and  $S(x \otimes p_g) = (g^{-1} \cdot x^{-1}) \otimes p_{g^{-1}}$  for all  $x \in L$ ,  $g \in G$ . This is termed the *smash coproduct* of  $kL$  and  $k[G]$ , written  $A = kL \bowtie k[G]$ .

Since  $\{1 \otimes p_g \mid g \in G\}$  is a set of orthogonal central idempotents in  $A$ , any  $A$ -module  $M$  decomposes as a direct sum,

$$M = \bigoplus_{g \in G} M_g,$$

where  $M_g = (1 \otimes p_g) \cdot M$ . Note that each component  $M_g$  is itself a  $kL$ -module by restriction of action to the subalgebra  $kL \cong kL \otimes k$  of  $A$ . By [13, Theorem 2.1], for any two  $A$ -modules  $M, N$ ,

$$(5.1) \quad (M \otimes N)_g \cong \bigoplus_{\substack{a, b \in G \\ ab=g}} M_a \otimes ({}^a N_b)$$

as  $kL$ -modules, where  ${}^a N_b$  is the conjugate  $kL$ -module that has underlying vector space  $N_b$  and action  $x \cdot_a n = (a^{-1} \cdot x) \cdot n$  for all  $x \in L$ ,  $n \in N$ .

As an algebra,  $A$  is a tensor product of  $kL$  and  $k[G]$ , and so its Hochschild cohomology is

$$\mathrm{HH}^*(A) \cong \mathrm{HH}^*(kL) \otimes \mathrm{HH}^*(k[G]) \cong \mathrm{HH}^*(kL) \otimes k[G].$$

The latter isomorphism occurs since  $k[G]$  is semisimple; its Hochschild cohomology is concentrated in degree 0 where it is isomorphic to the center of the commutative algebra  $k[G]$ . Let

$$H = H^{ev}(L, k) \otimes k[G],$$

a subalgebra of  $\mathrm{HH}^*(A)$  via the embedding of  $H^{ev}(L, k)$  into  $\mathrm{HH}^*(kL)$  discussed in Example 3.2. Then  $H$  satisfies (fg1) and (fg2) since  $H^*(L, k)$  does, and we use  $H$  to define support varieties for  $A$ -modules. The maximal ideal spectrum of  $H$  is

$$\mathrm{Max}(H) \cong \mathrm{Max}(H^{ev}(L, k)) \times G.$$

Define support varieties of  $kL$ -modules via  $H^{ev}(L, k)$  as usual (see Example 3.2), and denote such varieties by  $V_L$  in order to distinguish them from varieties for the

related  $A$ -modules. By (5.1), Proposition 3.3(ii), and the tensor product property for finite groups,

$$(5.2) \quad V_A((M \otimes N)_g) = \bigcup_{\substack{a, b \in G \\ ab=g}} (V_L(M_a) \cap V_L({}^a N_b)) \times g$$

for each  $g \in G$ . This formula gives a positive answer to Question 4.1(2'), yet it implies that the answers to Questions 4.1(2), (2''), (3), and (4) are no for some choices of  $L$  and  $G$ : For example, let  $p = 2$  and  $L = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , generated by  $g_1$  and  $g_2$ . Let  $G = \mathbb{Z}/2\mathbb{Z}$ , generated by  $h$ , acting on  $L$  by interchanging  $g_1$  and  $g_2$ . In this case  $H^*(L, k) \cong k[y_1, y_2]$  with  $y_1, y_2$  of degree 1, and so  $V_L(k)$  may be identified with affine space  $k^2$ . Let  $U = kL/(g_2 - 1)$ , a  $kL$ -module. Then  ${}^h U \cong kL/(g_1 - 1)$ . Note that  $V_L(U)$  may be identified with the line  $y_1 = 0$  and  $V_L({}^h U)$  may be identified with the line  $y_2 = 0$ . Now let  $M = U \otimes kp_h$  and  $N = U \otimes kp_1$ , with  $A = kL \otimes k[G]$  acting factorwise. By the tensor product formula (5.2),  $V_A(N \otimes M)$  consists of the line  $y_1 = 0$  paired with the group element  $h$ , while  $V_A(M \otimes N)$  has dimension 0. Thus the answers to Questions 4.1(2) and (2'') are no. By Proposition 3.3,  $M \otimes N$  is projective while  $N \otimes M$  is not. Similarly,  $M \otimes M$  is projective while  $M$  is not. More such examples are in [13], including examples showing that for any positive integer  $n$ , it can happen that  $M^{\otimes n}$  is projective while  $M^{\otimes(n-1)}$  is not, and examples of modules  $M$  for which  $V_A(M^*) \neq V_A(M)$ . These examples are generalized in [42] with Plavnik to crossed coproducts  $kL \bowtie_\sigma^\tau k[G]$  whose algebra and coalgebra structures are twisted by cocycles  $\sigma, \tau$ .

The above examples are all in positive characteristic. There are characteristic 0 examples in [42] that are completely analogous, where the group algebra  $kL$  is replaced by a quantum elementary abelian group as in Example 2.5. Again we find modules whose tensor product in one order is projective while in the other order is not, and nonprojective modules with a projective tensor power. In fact, these types of examples are very general, as the following theorem shows.

**Theorem 5.1.** [42] *Let  $A$  be a finite dimensional nonsemisimple Hopf algebra satisfying (fg1), (fg2), and the tensor product property. Then  $A$  is a subalgebra of a Hopf algebra  $K$  satisfying (fg1) and (fg2) for which the tensor product property does not hold. Moreover, there are nonprojective  $K$ -modules  $M, N$  for which  $M \otimes M$  and  $M \otimes N$  are projective, while  $N \otimes M$  is not projective.*

One such Hopf algebra is a smash coproduct  $(A \otimes A) \bowtie k^{\mathbb{Z}/2\mathbb{Z}}$  where the nonidentity element of the group  $\mathbb{Z}/2\mathbb{Z}$  interchanges the two tensor factors of  $A$ . See [42] for details. This is a Hopf algebra for which the answer to Question 4.1(1) is yes, and so it has a reasonable support variety theory and Question 4.1(2') has a positive answer. However the answers to Questions 4.1(2), (2''), (3), and (4) are no, just as in our earlier classes of examples in this section.

The positive answers in Section 4 to Questions 4.1 and the negative answers in this section all point to a larger question: What properties of a Hopf algebra ensure positive (respectively, negative) answers to Questions 4.1?

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